Exercise 20

Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is given by the vector function $\mathbf{r}(t)$.

$$\mathbf{F}(x, y, z) = (x + y^2)\,\mathbf{i} + xz\,\mathbf{j} + (y + z)\,\mathbf{k},$$
$$\mathbf{r}(t) = t^2\,\mathbf{i} + t^3\,\mathbf{j} - 2t\,\mathbf{k}, \quad 0 \le t \le 2$$

Solution

With the given parameterization in t, the line integral becomes

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt \\
&= \int_0^2 \langle x(t) + [y(t)]^2, x(t)z(t), y(t) + z(t)\rangle \cdot \frac{d}{dt}\langle t^2, t^3, -2t\rangle\,dt \\
&= \int_0^2 \langle t^2 + (t^3)^2, (t^2)(-2t), t^3 + (-2t)\rangle \cdot \langle 2t, 3t^2, -2\rangle\,dt \\
&= \int_0^2 \langle t^2 + t^6, -2t^3, t^3 - 2t\rangle \cdot \langle 2t, 3t^2, -2\rangle\,dt \\
&= \int_0^2 [(t^2 + t^6)(2t) - 2t^3(3t^2) + (t^3 - 2t)(-2)]\,dt \\
&= \int_0^2 (4t - 6t^5 + 2t^7)\,dt \\
&= \left(2t^2 - t^6 + \frac{1}{4}t^8\right)\Bigg|_0^2 \\
&= 2(2)^2 - (2)^6 + \frac{1}{4}(2)^8 \\
&= 8.
\end{aligned}$$